Extending Fermüller-style dialogues to classical logic

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Outline

Dialogical logic

- Introduced in the 1950s and 1960s to give an alternative semantics for intuitionistic logic.
- Based on the existence of winning strategies in finitary open two-person zero-sum games between Proponent (**P**) and Opponent (**O**).
- Players attack and defend formulas asserted by the other according to *particle* and *structural* rules.
- Meaning of connectives is given by their use.
- Recently extended to give new semantics for classical logic, modal logic, free logic, connexive logic, relevance logic, and others.

Motivation

- A soundness and completeness proof, for classical logic, of winning strategies for dialogue game seems to be difficult to find.
- Fermüller: uses a hypersequent formulation of classical logic and parallel dialogue games.
- Sørensen and Urzyczyn (second half of yesterday's talk): use a non-standard, specially-tuned sequent calculus that essentially mirrors Felscher-style dialogue games.
  - The rules have side conditions that are built directly from the dialogue rules for attacks and defenses of formulas.
  - "derivations in [**LKD**] are practically identical to winning proponent strategies".

Reference

M. H. Sørensen and P. Urzyczyn, "Sequent Calculus, Dialogues, and Cut Elimination", in Barendsen, E. et al. (eds.), *Reflectons on Type Theory, λ-Calculus, and the Mind: Essays Dedicated to Henk Barendregt on the Occasion of His 60th Birthday*, pp. 253–261.

New!

- First half of Sørensen's talk yesterday: a sequent calculus for classical logic suited to classical dialogues.
- Sørensen massages standard classical sequent calculus rules to fit the form of winning strategies.
- Goal: make winning strategies and sequent calculus derivations more or less isomorphic.
- Sørensen's classical games are similar to ours.
- By contrast, we use a standard sequent calculus for classical logic.

Results

Our two main results are:

Theorem (Soundness)

If **P** *has a winning strategy in the classical Fermüller-style dialogue game for* $\Pi \vdash \Delta$*, then* $\Pi \Rightarrow \Delta$ *is classically valid.*

Theorem (Completeness)

If $\Pi \Rightarrow \Delta$ *is classically valid, then* **P** *has a winning strategy in the classical Fermüller-style dialogue game for* $\Pi \vdash \Delta$*.*

Setup

Propositional language: formulas are built from atoms, $\bot$, and $\vee$, $\wedge$, and $\rightarrow$.

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Three *symbolic attack* expressions: ?, $\wedge_L$, and $\wedge_R$.

Two types of rules:

Particle rules: Govern how statements can be attacked and defended depending on their main connective.

Structural rules: Define what sequences of attacks and defenses count as dialogues.

Particle rules

Assertion	Attack	Response
p (atom)	?	—
$\varphi \wedge \psi$	$\wedge_L$	φ
	$\wedge_R$	ψ
$\varphi \vee \psi$	?	φ or ψ
$\varphi \to \psi$	φ	ψ

Table: Particle rules for Fermüller-style dialogue games

Structural rules Ei for IL

Definition (Ei structural rules)

- Start: **O** begins the game by attacking the initially disputed formula.
- Alternation: Moves strictly alternate between players **O** and **P**.
- Atom: Atomic formulas, including $\bot$, may be stated by both players, but only **O** can attack them.
- D11: If it is X's turn and there are multiple undefended attacks by Y that X, only the most recent one may be defended.
- D12: Any attack may be defended at most once.
- E: Each move of **O** (except the first) reacts directly to the immediately preceding move by **P**.

Reference

C. Fermüller, "Parallel Dialogue Games and Hypersequents for Intermediate Logics", *TABLEAUX 2003*, Mayer, M. C. and Pirri, F. (eds.), pp. 48–64.

Fermüller-style games for intuitionistic logic

- A **dialogue sequent** is a judgment of the form $\Pi \vdash \varphi$, where Π is a multiset of formulas.
- Moves in the game are state transitions $\Pi \vdash \Delta \longmapsto \Pi' \vdash \Delta'$ from one dialogue sequent to another.
- The multiset Π records the **granted formulas**: formulas asserted by **O**.

Definition (Active formula)

The formula φ is the **active formula** at a dialogue state if either **O** must attack φ next or φ is the last formula asserted by **P** that has been attacked by **O**.

- Winning conditions: the game ends with a win for **P** at a dialogue sequent $\Pi \vdash \varphi$ if either:
  - W: (*ipse dixisti*) **O** has just attacked a formula that he has granted earlier, or **O** grants a formula that was attacked earlier, or
  - $W_\perp$: $\perp \in \Pi$ (**O** grants $\perp$)

Example

Example (An Ei-game for $\vdash \varphi := (A \wedge (A \to B)) \to B$)

#		Move	Stance, Ref.	Granted Formulas		Active Formula
1	**O**	$A \wedge (A \to B)$	attack φ	$A \wedge (A \to B)$	$\vdash$	φ
2	**P**	$\wedge_L$	attack 1	$A \wedge (A \to B)$	$\vdash$	φ
3	**O**	A	defend 2	$A \wedge (A \to B), A$	$\vdash$	φ
4	**P**	$\wedge_R$	attack 1	$A \wedge (A \to B), A$	$\vdash$	φ
5	**O**	$A \to B$	defend 4	$A \wedge (A \to B), A, A \to B$	$\vdash$	φ
6	**P**	A	attack 5	$A \wedge (A \to B), A, A \to B$	$\vdash$	φ
7	**O**	?	attack 6	$A \wedge A \to B, A, A \to B$	$\vdash$	A

Adequacy of Fermüller-style games for IL

The notion of **winning strategy** for Fermüller-style dialgue games is defined in the usual way.
Main results:

Theorem (Soundness)

If there is a winning strategy for φ, then φ is intuitionistically valid.

Theorem (Completeness)

If φ is intuitionistically valid, then there is a winning strategy for φ.

The notion of intuitionistic validity is given by a "proof search" variant $\mathbf{LI}'$ of the standard sequent calculus **LI** for intuitionistic logic.

Extensions of Fermüller-style games

- Fermüller goes on to define **parallel dialogue games** and establishes correspondences between winning strategies for parallel dialogue games and hypersequent formulations of various **intermediate logics** (i.e., logics between IL and CL).
- One of the intermediate logics that Fermüller treats is the strongest one, CL.
- Result: a dialogical characterization of CL in terms of parallel dialogue games, when CL is understood as a certain hypersequent calculus.
- Our contribution: an extension of non-parallel Fermüller-style games and a correspondence between winning stategies for them and a non-hypersequent formulation of CL.

Reference

Alama, J. and Knoks, A. and Uckelman, S. L., "Dialogue games for classical logic". Preprint available on our homepages.

Fermüller-style dialogue games for classical logic

- A **classical dialogue sequent** is a judgment of the form $\Pi \vdash \Delta$, where Π and Δ are multisets.
  - As before, members of Δ are called **granted fomulas** and record the assertions made by **O**.
  - Members of Δ are called **disputed formulas** and record formulas *attacked* by **O**.
- We do not need the notion of active formula.
- Winning conditions: The game ends at a dialogue state $\Pi \vdash \Delta$ with a win for **P** if:
  - W^{CL}: $\Pi \cap \Delta$ is non-empty (*ipse dixisti*: some formula is both contested and granted), or
  - $W_{\perp}$: $\perp \in \Pi$ (**O** granted $\perp$).

Structural rules for classical logic

Definition (CL structural rules)

Start The first move of the dialogue is carried out by **O** and consists in an attack on (the unique) initially disputed formula φ.

Alternation Moves strictly alternate between players **O** and **P**.

Atom Atomic formulas, including $\bot$, may be stated by both players, but only **O** can attack them.

E Each (but the first) move of **O** reacts directly to the immediately preceding move by **P**.

These are the rules Ei for intuitionistic logic from before, minus $\mathrm{D11}$ and $\mathrm{D12}$.

As before, **P** can attack formulas in Π at any time.

In addition, **P** can assert formulas in Δ as defenses at any time.

Example 1: Dummett's formula

Example (An Ei-game for $\vdash \varphi := (A \to B) \vee (B \to A)$)

#		Move	Stance, Ref.	Granted Formulas		Disputed Formulas
1	**O**	?	attack φ		$\vdash$	φ
2	**P**	$A \to B$	defend 1		$\vdash$	φ
3	**O**	A	attack 1	A	$\vdash$	$\varphi, A \to B$
4	**P**	B	defend 3	A	$\vdash$	$\varphi, A \to B$
5	**O**	?	attack 4	A	$\vdash$	$\varphi, A \to B, B$
6	**P**	$B \to A$	defend	A	$\vdash$	$\varphi, A \to B, B$
7	**O**	B	attack 6	A, B	$\vdash$	$\varphi, A \to B, B, A$

The game ends here with a win for **P**, thanks to winning condition W^{CL}, because the left-hand side and rigt-hand side of the dialogue sequent after **O**'s move in step 7 intersect.

Example 2: Double negation elimination

Example (An Ei-game for $\vdash \varphi := \neg\neg A \to A$)

#		Move	Stance, Ref.	Granted Formulas		Disputed Formulas
1	**O**	$\neg\neg A$	attack φ	$\neg\neg A$	$\vdash$	φ
2	**P**	$\neg A$	attack 1	$\neg\neg A$	$\vdash$	φ
3	**O**	A	attack 2	$A, \neg\neg A$	$\vdash$	$\varphi, \neg A$
4	**P**	A	defend φ	$A, \neg\neg A$	$\vdash$	$\varphi, \neg A$
5	**O**	?	attack 4	$A, \neg\neg A$	$\vdash$	$\varphi, \neg A, A$

Classical sequent calculus

- We employ a "proof search" variant **GKcp**′ of a standard calculus **GKcp** for CL.
- Derived objects are sequents $\Delta \Rightarrow \Pi$, where Δ and Π are multisets.
  - We restrict attention to only non-empty succedents Π.
  - This is no loss of generality because $\Pi \Rightarrow \emptyset$ is derivable in **GKcp**′ iff $\Pi \Rightarrow \perp$ is derivable in **GKcp**′.
- Our rules are "strictly cumulative" in the sense that the antecedent of every conclusion of every application of our rules is a subset of the antecedent of the hypotheses, and similarly for every succedent.
- The calculus **GKcp**′ is clearly sound and complete for CL. Definition

Reference

A. S. Troelstra and H. Schwichtenberg, *Basic Proof Theory*, 2nd edition, Cambridge University Press, 2000.

Soundness

Theorem

If **P** *has a winning strategy in the classical Fermüller-style dialogue game for* $\Pi \vdash \Delta$*, then* $\Pi \Rightarrow \Delta$ *is classically valid.*

- The proof goes by induction on the depth of a winning strategy τ for $\Pi \vdash \Delta$.
- We show that for every **P**-node of τ (that is, a node of τ representing a state of the game where it is **P**'s turn to move) that there exists a winning strategy for the dialogue sequent associated with the node.

Completeness

Theorem

If $\Pi \Rightarrow \Delta$ is classically valid, then **P** *has a winning strategy in the classical Fermüller-style dialogue game for $\Pi \vdash \Delta$.*

- Like Fermüller, we restrict attention to strongly analytic **GKcp**′-deductions (i.e., those that do not contain any application of any structural rule).
  - This is no loss of generality: $\Pi \Rightarrow \Delta$ is provable in **GKcp**′ iff there is a strongly analytic **GKcp**′-deduction of $\Pi \Rightarrow \Delta$.
- The proof is by structural induction on all strongly analytic **GKcp**′-deductions.

Conclusion

- We designed a dialogue system that is specially tuned to classical logic, but our notion of classical validity is a standard sequent calculus.
- It is not clear to us yet how our dialogues relate to Fermüller's characterization of classical logic using parallel dialogue games, or to Sørensen's approach from yesterday.

The calculus GKcp′

Axioms: $\varphi, \Pi \Rightarrow \Delta, \varphi$ and $\bot, \Pi \Rightarrow \Delta$

$$\vee\text{L}: \quad \frac{A \vee B, A, \Pi \Rightarrow \Delta \qquad A \vee B, B, \Pi \Rightarrow \Delta}{A \vee B, \Pi \Rightarrow \Delta}$$

$$\vee\text{R}: \quad \frac{\Pi \Rightarrow \Delta A \vee B, A, B}{\Pi \Rightarrow \Delta, A \vee B}$$

$$\wedge\text{L}: \quad \frac{A \wedge B, A, B, \Pi \Rightarrow \Delta}{A \wedge B, \Pi \Rightarrow \Delta}$$

$$\wedge\text{R}: \quad \frac{\Pi \Rightarrow \Delta, A \wedge B, A \qquad \Pi \Rightarrow \Delta, A \wedge B, B}{\Pi \Rightarrow \Delta, A \wedge B}$$

$$\to\text{L}: \quad \frac{A \to B, \Pi \Rightarrow \Delta, A \qquad A \to B, B, \Pi \Rightarrow \Delta}{A \to B, \Pi \Rightarrow \Delta}$$

$$\to\text{R}: \quad \frac{A, \Pi \Rightarrow \Delta, A \to B, B}{\Pi \Rightarrow \Delta, A \to B}$$

Literature

- C. Fermüller, "Parallel Dialogue Games and Hypersequents for Intermediate Logics", *TABLEAUX 2003*, Mayer, M. C. and Pirri, F. (eds.), pp. 48–64.
- M. H. Sørensen and P. Urzyczyn, "Sequent Calculus, Dialogues, and Cut Elimination", in Barendsen, E. et al. (eds.), *Reflectons on Type Theory, λ-Calculus, and the Mind: Essays Dedicated to Henk Barendregt on the Occasion of His 60th Birthday*, pp. 253–261.
- Alama, J. and Knoks, A. and Uckelman, S. L., "Dialogue Games for Classical Logic". Preprint available on our homepages.