

A curious dialogical logic and its composition problem

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- Introduced in the 1950s and 1960s to give an alternative semantics for intuitionistic logic.
- Based on the existence of winning strategies in finitary open two-person zero-sum games between Proponent and Opponent.
- Players attack and defend formulas asserted by the other according to *particle* and *structural* rules.
- Meaning of connectives is given by their use.
- Recently extended to give new semantics for classical logic, modal logic, free logic, connexive logic, relevance logic, and others.

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Two types of rules:

Particle rules Govern how statements can be attacked and defended depending on their main connective.

Structural rules Define what sequences of attacks and defenses count as dialogues.

Assertion	Attack	Response
$\varphi \wedge \psi$	\wedge_L	φ
	\wedge_R	ψ
$\varphi \vee \psi$?	φ or ψ
$\varphi \rightarrow \psi$	φ	ψ
$\neg\varphi$	φ	—

Table: Particle rules for dialogue games

Structural rules

- (D10) P may assert an atomic formula only after it has been asserted by O before.
- (D11) If p is a P -position, and if at round $n - 1$ there are several open attacks made by O , then only the latest of them may be answered at n (and the same with P and O reversed).
- (D12) An attack may be answered at most once.
- (D13) A P -assertion may be attacked at most once.
 - (E) O can react only upon the immediately preceding P -statement.

Reference

W. Felscher, "Dialogues, Strategies, and Intuitionistic Provability", *Annals of Pure and Applied Logic* 28 (1985): 217–254.

Winning plays and winning strategies

Play alternates between Proponent and Opponent (starting with Proponent at move 0), and every move (except the initial assertion) is either an attack or a defense against some earlier assertion.

Definition

Given a set S of structural rules, an S -dialogue for a formula φ is a dialogue commencing with φ that adheres to the rules of S . Proponent *wins* an S -dialogue if there is a round where Opponent has no legal moves available.

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Remark: According to this definition, if the dialogue *can* go on, then neither player is said to win; the game proceeds as long as moves are available.

Definition

A player has *winning strategy* for a formula φ if no matter what move the other player makes, the first player has a legal move, and eventually the second player has no legal moves left.

Definition

For a set S of dialogue rules and a formula φ , the relation $\vDash_S \varphi$ means that Proponent has an S -winning strategy for φ . If $\not\vDash_S \varphi$, then we say that φ is S -invalid.

Theorem (Felscher)

- A formula φ is intuitionistically valid iff $\vDash_D \varphi$.
- A formula φ is intuitionistically valid iff $\vDash_{D+E} \varphi$.

Theorem (Folklore?)

A formula φ is classically valid iff $\vDash_{D10+D13+E} \varphi$.

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- In what ways can the structural rules be changed?
 - Remove rules from a rule-set known to give rise to a logic.
 - Create and add new rules.
- In what cases will the resulting rule-set actually give you a logic?

What is a logic?

Definition

Given a language \mathcal{L} , a *logic* is a set L of \mathcal{L} -formulas which is closed under modus ponens (that is, if $\varphi \in L$ and $\varphi \rightarrow \psi \in L$, then $\psi \in L$ as well).

Note

Note that we deviate from Tarski's definition of logic by not requiring that L be closed under unrestricted uniform substitution.

Definition

The *composition problem* for a set of dialogue rules S asks whether the set S of formulas φ for which Proponent has a winning strategy in the S -dialogue game commencing with φ is closed under modus ponens.

A positive solution to the composition problem for a ruleset S shows that S generates a logic (in the aforementioned sense of the term); a negative solution shows that the set of S -valid formulas lacks logical coherence.

What happens when we remove structural rules?

Let $D := D10 + D11 + D12 + D13$. We have:

$$D + E = IL = D.$$

Thus, adding the E rule to the D ruleset has no effect on the generated logic, IL .

Question: is the same true for classical logic? That is:

$$D10 + D13 + E = CL = D10 + D13?$$

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Definition

Let $N = D10 + D13$. The logic N is the set of formulas for which P has a winning N -strategy.

Some N-valid formulas

Many familiar formulas are valid in N:

$$\begin{array}{ll} p \vee \neg p & \neg p \vee \neg\neg p \\ (p \rightarrow q) \vee (p \rightarrow \neg q) & (p \rightarrow q) \vee (q \rightarrow p) \\ \neg\neg p \rightarrow p & p \rightarrow \neg\neg p \\ p \rightarrow (p \vee q) & p \rightarrow (p \wedge p) \\ \neg p \rightarrow (p \rightarrow q) & \neg(p \vee \neg p) \rightarrow q \end{array}$$

One can compute the validity of these formulas by hand by playing games for them. Better, visit our dialogue game web application at <http://dialogical-logic.info/>.

Properties of winning \mathbb{N} -strategies (1)

Theorem

Every branch for an \mathbb{N} -dialogue tree that contains a defensive move by O either terminates at an O -move, or is infinite.

Corollary

No \mathbb{N} -winning strategy contains a branch where O defends.

Lemma (Weakening)

If $\vDash_{\mathbb{N}} \psi$, then $\vDash_{\mathbb{N}} \varphi \rightarrow \psi$, for all formulas φ .

Properties of winning N-strategies (2)

Lemma

No atomic formula is N-valid.

Corollary

N is consistent.

Theorem (Characterization of N-valid negations)

If $\vDash_N \neg\varphi$, then φ is a negation $\neg\psi$ and $\vDash_N \psi$.

Characterization of N-valid implications

Theorem (Characterization of implication)

Every N-valid implication $\varphi \rightarrow \psi$ satisfies one of the following three conditions: (1) φ is atomic, (2) φ is negated, (3) $\vDash_N \psi$.

Proof.

Case (3) is just a restatement of the Weakening Lemma. Suppose now that φ is not atomic and ψ is not an N-validity. Proceed by cases:

- If φ is an implication $\alpha \rightarrow \beta$, then the N-dialogue tree opens with O attacking the initial statement by asserting $\alpha \rightarrow \beta$. In any N-winning strategy for $(\alpha \rightarrow \beta) \rightarrow \psi$, Proponent cannot attack O 's assertion of $\alpha \rightarrow \beta$. Thus, any winning strategy s must choose, for P 's response to O 's initial attack, to defend by asserting the consequent ψ of the entire formula, and no branch of s can attack the antecedent implication $\varphi \rightarrow \psi$. By renumbering the reference labels for nodes of s below the P 's assertion of ψ in the obvious way (renumber k to $k - 2$), we obtain a winning strategy for ψ , contradicting our assumption.
- Likewise, φ cannot be a disjunction, nor could it be a conjunction, for similar reasons: In any N-winning strategy s for $(\alpha \vee \beta) \rightarrow \psi$ (or for $(\alpha \wedge \beta) \rightarrow \psi$), Proponent never attacks $\alpha \vee \beta$ (respectively, $\alpha \wedge \beta$), so we can recover from s a winning strategy for ψ , contradicting our assumption.

The only possibility left is that φ is a negation. □

From the characterization of implication it is straightforward to prove the following:

Theorem (Composition)

If $\vDash_N \varphi$ and $\vDash_N \varphi \rightarrow \psi$, then $\vDash_N \psi$.

Properties of N (1)

Theorem

$N \subset CL$.

Proof.

Every D10 + D13-strategy is also a D10 + D13 + E-strategy, and the latter ruleset captures CL.

That the inclusion is strict follows from the fact that

$\not\models_N (((p \rightarrow q) \rightarrow p) \rightarrow p)$ (Peirce's formula), which is classically valid. □

Although Peirce's formula is not valid in N, curiously, both excluded middle $p \vee \neg p$ and double negation elimination $\neg\neg p \rightarrow p$ are valid in N.

Properties of N (2)

Theorem

$N \not\subseteq IL$ and $IL \not\subseteq N$.

Proof.

For the first claim, $\vDash_N p \vee \neg p$. For the second claim, $\vDash_{IL} (\neg p \vee \neg q) \rightarrow \neg(p \wedge q)$, which, by Theorem 13 is not N-valid, since $\not\vdash_N \neg(p \wedge q)$. □

Since N is neither sub-intuitionistic nor super-intuitionistic, but is sub-classical, it lies in an interesting and as yet under-investigated part of the lattice of propositional logics.

Properties of N (3)

Theorem

N is not closed under uniform substitution, though it is consistent with uniform substitution.

Proof.

The characterization of N-valid implications can be used to produce many failures of uniform substitution. Thus, $p \rightarrow p$ is N-valid, but $(p \wedge p) \rightarrow (p \wedge p)$ is not.

Closing N under uniform substitution results in a consistent set: any witness to a contradiction in the closure gives rise to a contradiction in N. □

Depending on one's definition of "logic", the failure of uniform substitution for N may detract from calling N a logic, its closure under modus ponens notwithstanding. One can instead work with the closure of N under uniform substitution; many curiosities remain.

Conclusion

- We made a simple and intuitive modification of the usual rules for classical dialogue games, and obtained a set N of dialogically valid formulas for which we proved a positive answer for its composition problem, thus allowing us to call N a logic.
- The positive solution was proved directly, rather than detouring through a cut-free proof system. (We lack a proof theory for N .)
- The logic N has a number curious features which arise from the fact that if Opponent can defend once, he can always defend again.
- Philosophical questions: What kind of interpretation can N be given? If the meaning of logical connectives is given by their use (that is, how they can be attacked and defended), how is it that keeping the particle rules fixed and but the structural rules slightly can result in such a strange logic? Do logics like N undermine Lorenzen's "meaning as use" interpretation of dialogical logic?