Exploring Steinitz-Rademacher polyhedra

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A theory of three-dimensional polyhedra

- *Vorlesungen über die Theorie der Polyeder*, a basic work in graph theory by E. Steinitz (completed with the help of H. Rademacher).
- Fundamental problem: *when is a graph realizable as a convex figure?*
- Result: a graph g is isomorphic to the 1-skeleton of a convex real polyhedron iff g is planar and 3-connected.
- Side result: a simple first-order theory of abstract three-dimensional polyhedra.

A limitation of FOL

First-order logic cannot express some of the simple properties of polyhedra:

- connectedness
- having an even number of vertices/edges/faces
- satisfying Euler's formula
- etc.

Moving to monadic second-order logic can help (recent work by Bruno Courcelle), but its expressiveness is also limited.

However: using FOL brings powerful ATP systems to bear on geometry.

Axioms of SR

Signature: one binary incidence relation I, three sorts V (vertex), E (edge), and F.

- there are vertices, edges, and faces;
- every element is a vertex, edge, or a face;
- I is symmetric;
- no two vertices are incident, and the same goes for edges and faces;
- if $V(v)$, $E(e)$, $F(f)$, $I(v,e)$ and $I(e,f)$, then $I(v,f)$;
- every edge is incident with exactly two vertices;
- every edge is incident with exactly two faces;
- $V(v)$, $F(f)$ and $I(v,f)$ imply that there are exactly two edges incident with both v and f; and
- every vertex and every face is incident with at least one other element.

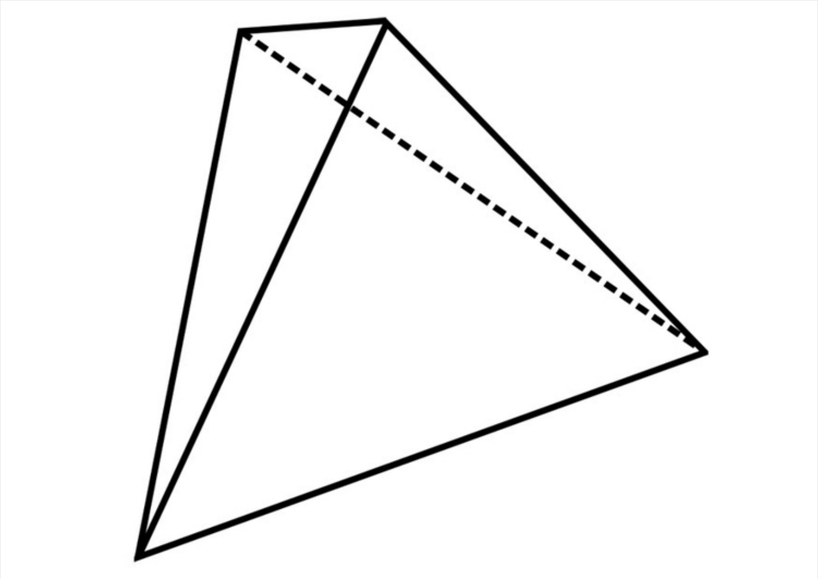
Problems about SR

- Is SR consistent? What is its smallest model?
- For which natural numbers k is SR k-categorical? (If not, how many non-isomorphic models are there?)
- Can one recover well-known polyhedra as models of SR?
- Can we discover unusual or unexpected SR-polyhedra?

Many of these problems, even for 'small' or 'simple' polyhedra, give rise to challenging problems for ATP systems.

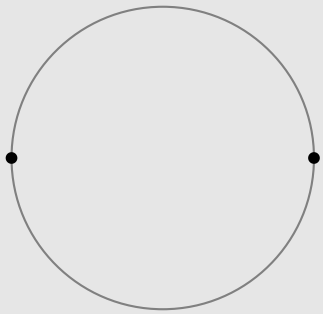
Consistency of SR

Nearly any polyhedron that comes to mind is likely to be an SR-polyhedron:



Small models of SR (I)

The smallest SR-polyhedron M_6: 2 vertices, 2 edges, 2 faces:



By searching through all models from size 1 to 5 without finding any, **mace4** demonstrates that M_6 is indeed the smallest model of SR.

Small models of SR (II)

Theorem: M_6 is the *only* model of SR of cardinality 6.

Proof: Each of the 28 triples (N_0, N_1, N_2) that sum to 6 generates an extension $\mathrm{SR}_{N_0,N_1,N_2}$ of SR, each saying that there exist exactly N_0 vertices, N_1 edges, and N_2 faces.

All but one of these 28 first-order theories is inconsistent, namely the one associated with $(2,2,2)$. And we know $\mathrm{SR}_{2,2,2}$ is consistent.

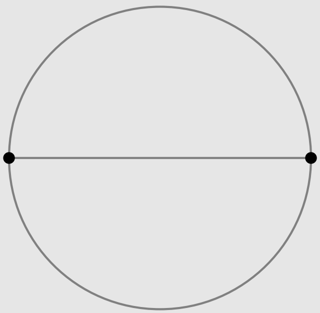
Small models of SR (III)

To show that M_6 is the only model of cardinality 6 having 2 vertices, 2 edges, and 2 faces, consider the formula φ defined as

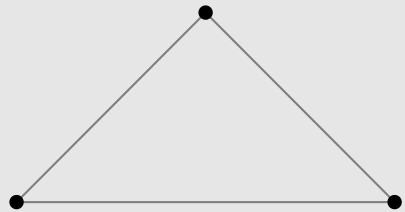
$$\exists x_0, \ldots, x_5 \left[\begin{array}{c} V(x_0) \wedge V(x_1) \wedge x_0 \neq x_1 \wedge \forall x\, (V(x) \rightarrow (x = x_0 \vee x = x_1)) \\ \wedge \\ E(x_2) \wedge E(x_3) \wedge x_2 \neq x_3 \wedge \forall x\, (E(x) \rightarrow (x = x_2 \vee x = x_3)) \\ \wedge \\ F(x_4) \wedge F(x_5) \wedge x_4 \neq x_5 \wedge \forall x\, (F(x) \rightarrow (x = x_4 \vee x = x_5)) \\ \wedge \\ \neg \left[\begin{array}{c} I(x_0,x_2) \wedge I(x_1,x_2) \wedge I(x_0,x_3) \wedge I(x_1,x_3) \\ \wedge \\ I(x_2,x_4) \wedge I(x_3,x_4) \wedge I(x_2,x_5) \wedge I(x_3,x_5) \end{array} \right] \end{array} \right]$$

Small models of SR (IV)

(At least) two models of cardinality 8:



2 vertices, 3 edges, 3 faces

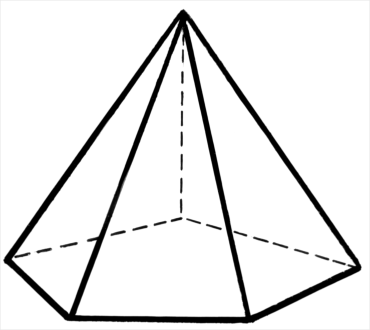


3 vertices, 3 edges, 2 faces

(The existence of the right-hand model follows from the duality of SR under the exchange of the predicates V and F.)

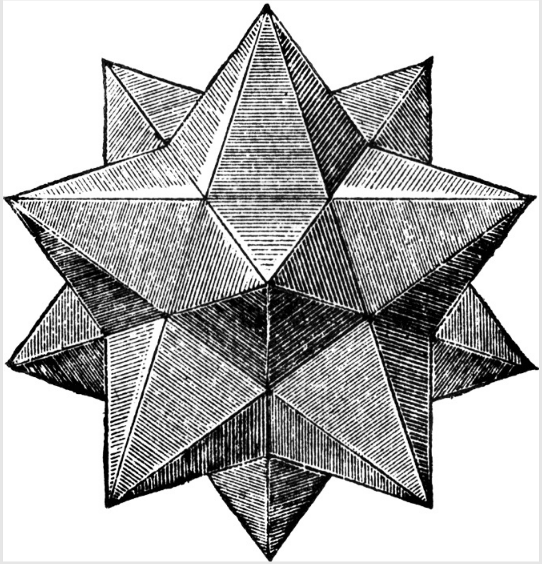
‘Large’ models

- Larger models of SR can be found (less than 10 seconds for **paradox** on models up to cardinality 16), but this is arguably only the beginning of when really interesting polyhedra start to show up.
- The tetrahedron and some pyramids can be recovered by **paradox**, but larger models seem to be out of reach.



‘Large’ models (II)

A very large model: cardinality 240! Can we find this stellated dodecahedron?



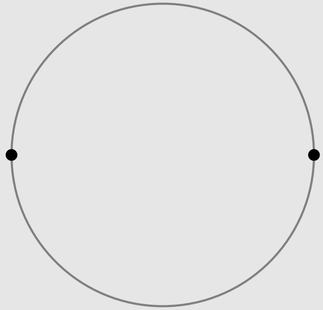
‘Large’ models (III)

Recovering known polyhedra as ‘large’ models of SR is already challenging enough. Consider further:

- The cube and the octahedron are dual platonic solids, both having cardinality 26; are they the only SR-polyhedra of cardinality 26, or are there others?
- The dodecahedron and the icosahedron are dual platonic solids, both having cardinality 62; are they the only SR-polyhedra of cardinality 62, or are there others?
- Can one find other models like the stellated dodecahedron (extremely large cardinality)?

Extensional theory (I)

Do we really want to accept



as a 'polyhedron'?

Add extensionality to SR to rule out, say, pairs of vertices that are the 'end-points' of two distinct edges.

Extensional theory (II)

- for all vertices v_1 and v_2: if v_1 and v_2 are incident with the same edges, then $v_1 = v_2$;
- for all vertices v_1 and v_2: if v_1 and v_2 are incident with the same faces, then $v_1 = v_2$;
- for all edges e_1 and e_2: if e_1 and e_2 are incident with the same vertices, then $e_1 = e_2$.
- etc.

(There are six principles altogether.) Adding these principles to SR makes the theory more geometrically interesting: the tetrahedron (cardinality 14) becomes the smallest model.

Extensional theory (III)

Although extensional SR has more geometric content, investigating it with ATP tools is more challenging: several new skolem functions are required.

In the extensional context:

- most platonic solids seem to be beyond reach;
- determining categoricity or counting the number of models is likewise quite challenging in the presence of extensionality.

Future work

- Tools used so far: **mace4**/**prover9**, **paradox**. These are tools for working with *unsorted* theories. Clearly, our domain is naturally understood as sorted. Sorted tools (especially sorted model finders) are more appropriate.
- Constraint solving methods (e.g., those behind **sem**).
- Manual/custom reduction, in the case of model-finding problems, to SAT.
- Add these problems to the TPTP.